Multivariable analysis 2021-2022
Exam, Wednesday 26 January, 16:00-18:00

- Below you can find the exam questions. The number of points per question is indicated in a box. There are 4 questions summing up to 90 points, and you get 10 points for free.
- Please provide proper arguments when writing your solutions. Answers not accompanied by explanations do not count as such. At the same time, try not to overdo it. You may use all of the results covered in this course, you don't have to justify them separately. It suffices to give a (correct) statement and apply it to your question.
- When handing in your solutions, please do not forget to write your name and student number on the envelope.
- Good luck!

Solutions

1. $\boxed{8+12 = 20 \text{ pts}}$ Verify that the following maps are two times differentiable at the origin $O \in \mathbb{R}^2$ and compute their first and second differentials at O (in matrix form):

   i) $F(x, y) = (x - y, \sqrt{x^6 + y^6})\colon \mathbb{R}^2 \to \mathbb{R}^2$;

   ii) $f(x, y)\colon \mathbb{R}^2 \to \mathbb{R}$, where the function f is given by

$$f(x, y) = \int_0^x \int_0^y e^{t^2+s^2} dtds;$$

i) $F = (f_1, f_2) = (x - y, \sqrt{x^6 + y^6})$ is k times differentiable if and only if its components f_1 and f_2 are k times differentiable.

Since $f_1 = x - y$ is linear, it is two times differentiable (everywhere).

The second component $f_2 = \sqrt{x^6 + y^6}$ has continuous first and second order partial derivatives, which shows that f_2 is C^2 on $\mathbb{R}^2$ and hence twice differentiable everywhere.

Moreover, at the origin, the first and second partial derivatives of f_2 are zero. Hence $Df_2|_{0,0}$ and $D^2 f_2|_{0,0}$ vanish.

Since $Df_1 = (1, -1)$ and $D^2 f_1$ vanishes, we get

$$DF|_{0,0} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

and $D^2F|_{0,0}$ is a zero matrix (of size 4×2 if one views $D^2F|_{0,0}$ as a linear map $D^2F|_{0,0}\colon \mathbb{R}^2 \to \mathcal{L}(\mathbb{R}^2,\mathbb{R}^2) \simeq \mathbb{R}^4$).

ii) Observe that $e^{t^2+s^2}$ is C^∞ (hence continuous). Hence

$$\frac{\partial f}{\partial x} = e^{x^2}\int_0^y e^{t^2}dt, \quad \frac{\partial^2 f}{\partial x^2} = 2xe^{x^2}\int_0^y e^{t^2}dt \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} = e^{x^2+y^2}.$$

Applying Fubini's theorem, we get that the remaining partial derivatives are obtained by interchanging x and y:

$$\frac{\partial f}{\partial y} = e^{y^2}\int_0^x e^{s^2}ds, \quad \frac{\partial^2 f}{\partial y^2} = 2ye^{y^2}\int_0^x e^{s^2}ds \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = e^{x^2+y^2}.$$

This shows that f is C^2 (even C^∞) and hence twice differentiable everywhere. Substituting $x = y = 0$, gives

$$\frac{\partial f}{\partial x}|_{0,0} = \frac{\partial f}{\partial y}|_{0,0} = 0, \quad \frac{\partial^2 f}{\partial x^2}|_{0,0} = \frac{\partial^2 f}{\partial y^2}|_{0,0} = 0, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x}|_{0,0} = 1.$$

Therefore, $Df|_{0,0} = (0,0)$, and

$$D^2f|_{0,0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2. 3+8+14 = 25 pts Consider the following linear differential equation

$$x'' + (\cos t)x' - (\sin t)x = 0. \tag{1}$$

i) Find a particular solution of (1) using the substitution $x = e^{c\sin t}$, $c \in \mathbb{R}$;

ii) Write (1) in matrix form

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = A(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and solve the differential equation $W' = \mathrm{Tr}(A(t))W$ for the Wronskian W;

iii) Use i) and ii) to find a fundamental system of independent solutions of (1).

Hint: recall that W (when non-zero) is the determinant of a fundamental matrix.

i) Substitution of $x = e^{c\sin t}$ into (1) gives $c = -1$, so $x = e^{-\sin t}$ is a particular solution.

ii) The equation in matrix form is obtained by setting $x_1 = x,\ x_2 = x'$. This gives $x_1' = x_2$ and $x_2' + (\cos t)x_2 - (\sin t)x_1 = 0$, which can be written as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \sin t & -\cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Hence $\operatorname{Tr}(A(t)) = -\cos t$ and $W' = \operatorname{Tr}(A(t))W$ takes the form $W' = -(\cos t)W$. We get that $W = be^{-\sin t}$, where b is an arbitrary constant, which is non-zero for independent solutions.

iii) Let $x(t) = e^{-\sin t}$ be the solution found in i) and $y(t)$ be an independent solution. Then

$$W = \det \begin{pmatrix} x(t) & y(t) \\ x'(t) & y'(t) \end{pmatrix}$$

is a Wronskian. Hence, from ii), we get the equation $x(t)y'(t) - y(t)x'(t) = be^{-\sin t}$. If $y(t)$ is independent from $x(t)$, then $b \neq 0$; therefore, without loss of generality, $b = 1$. The equation has thus the form

$$e^{-\sin t}y' + \cos t e^{-\sin t}y = e^{-\sin t} \iff y' + \cos t y = 1.$$

Solving the homogeneous equation $y' + \cos t y = 0$ gives $y = Ce^{-\sin t}$, where C is a constant (this is clear since $x = e^{-\sin t}$ solves this equation: $xx' - x'x = 0$).

To solve the inhomogeneous equation, we apply the variation of constants method. It gives the following equation on $C = C(t)$:

$$C'e^{-\sin t} = 1.$$

Hence $C = \int_0^t e^{\sin\tau} d\tau + C_0$. As a fundamental system of solutions we can thus take $x = e^{-\sin t}$ and $y = (\int_0^t e^{\sin\tau} d\tau)e^{-\sin t}$.

3. $\boxed{8+9 = 17 \text{ pts}}$ Let $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ be the standard volume form on $\mathbb{R}^3$. Recall that for a vector field v on $\mathbb{R}^3$, $i_v(\omega)$ is the two-form defined by

$$i_v(\omega)(u_1, u_2) = \omega(v, u_1, u_2).$$

In what follows, $v = \operatorname{grad} f$ with $f \colon \mathbb{R}^3 \to \mathbb{R}$ given by $f = x^2 + y^2 - z^2$.

i) Compute the differential two-form $i_{\operatorname{grad} f}(\omega)$;

ii) Compute the pull-back $h^\star(i_{\operatorname{grad} f}(\omega))$ under the map $h = (s\cos t, s\sin t, s) \colon \mathbb{R}^2 \to \mathbb{R}^3$.

i) Observe that $\mathrm{grad} f = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z}$. Since $\omega(\mathrm{grad} f, u_1, u_2)$ is the determinant of the matrix of components of v, u_1, u_2, expanding this matrix in the first row, gives

$$\omega(\mathrm{grad} f, u_1, u_2) = 2x\mathrm{d}y \wedge \mathrm{d}z(u_1, u_2) - 2y\mathrm{d}x \wedge \mathrm{d}z(u_1, u_2) - 2z\mathrm{d}x \wedge \mathrm{d}y(u_1, u_2).$$

Hence $i_{\mathrm{grad} f}(\omega) = 2x\mathrm{d}y \wedge \mathrm{d}z + 2y\mathrm{d}z \wedge \mathrm{d}x - 2z\mathrm{d}x \wedge \mathrm{d}y$.

ii) To compute the pull-back, it suffices to substitute in $2x\mathrm{d}y \wedge \mathrm{d}z + 2y\mathrm{d}z \wedge \mathrm{d}x - 2z\mathrm{d}x \wedge \mathrm{d}y$ the expressions of x, y, z in terms of s, t. This gives

$$h^\star(i_{\mathrm{grad} f}(\omega)) = 2s\cos t\mathrm{d}(s\sin t) \wedge \mathrm{d}s + 2s\sin t\mathrm{d}s \wedge \mathrm{d}(s\cos t) - 2s\mathrm{d}(s\cos t) \wedge \mathrm{d}(s\sin t) =$$
$$2s^2\cos^2 t\mathrm{d}t \wedge ds - 2s^2\sin^2 t\mathrm{d}s \wedge \mathrm{d}t - 2s(s\cos^2 t\mathrm{d}s \wedge \mathrm{d}t - s\sin^2 t\mathrm{d}t \wedge \mathrm{d}s) = 4s^2\mathrm{d}t \wedge \mathrm{d}s.$$

4. 8+10+10 = 28 pts Consider the following level set

$$M^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^4 = 1\}$$

and let the two-form ω be given by $\omega = x\mathrm{d}y \wedge \mathrm{d}z$.

i) Verify that M^2 is a regular orientable (C^∞-)smooth surface in $\mathbb{R}^3$ (i.e., M^2 is locally a graph of a smooth function, and M^2 admits a nowhere vanishing smooth two-form);
ii) Prove that the pull-back $g^\star(\omega)$ is closed, but not exact on M^2; (here g denotes the inclusion of M^2 in $\mathbb{R}^3$);
iii) Compute the integral $\int_{M^2} g^\star(\omega)$.

i) Observe that M^2 is a level set of the function $f = x^2+y^2+z^4 \colon \mathbb{R}^3 \to \mathbb{R}$. Clearly f is smooth, and $M^2 = \{f = 1\}$ is non-empty. Moreover, the gradient $\mathrm{grad} f = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 4z^3\frac{\partial}{\partial z} = (2x, 2y, 4z^3)$ is never vanishing on M^2. By the Implicit Function Theorem, M^2 is locally a graph of a smooth function and hence a regular smooth surface.

This surface is orientable since $g^\star(i_{\mathrm{grad} f}(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z))$ is a nowhere vanishing smooth two-form on M^2.

ii) Let $\bar{U}$ be the compact region enclosed by M^2; specifically, let $\bar{U} = \{f \leq 1\}$ and observe that $\partial U = M^2$. By Stokes's theorem,

$$\int_{M^2} g^\star(\omega) = \int_{\bar{U}} \mathrm{d}\omega.$$

But $\mathrm{d}\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is the volume form on $\mathbb{R}^3$. Since $\bar{U}$ contains a small ball $B_\varepsilon(0)$ around the origin,

$$\int_{\bar{U}} \mathrm{d}\omega > 0.$$

But if the form $g^\star(\omega)$ was exact, then the integral

$$\int_{M^2} g^\star(\omega)$$

would be zero (again by Stokes's, since $\partial M^2 = \varnothing$). Hence $g^\star(\omega)$ is not exact. It is closed as a top form on M^2.

iii) To compute integral $\int_{M^2} g^\star(\omega)$, we again apply Stokes's theorem as in part ii):

$$\int_{M^2} g^\star(\omega) = \int_{\bar{U}} \mathrm{d}\omega.$$

Since $\mathrm{d}\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, we thus need to compute the volume of $\bar{U}$. This can be done using Fubini's theorem as follows:

$$\int_{\bar{U}} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \int_{x^2+y^2+z^4 \leq 1} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z =$$

$$\int_{-1}^{1} \left(\int_{x^2+y^2 \leq 1-z^4} \mathrm{d}x\mathrm{d}y \right) \mathrm{d}z = \int_{-1}^{1} \pi(1 - z^4)\mathrm{d}z = 1.6\pi.$$

End of exam